Overview: The problems below build on concepts that should be familiar to you based on your course on atmospheric thermodynamics. Nonetheless, many of you may find the problems difficult. I suggest you work on them in groups. A useful approach might be to review the problems, thinking about how you would solve them alone, then if time permits in groups, prior to the first discussion session. Ideas for solving the problems can be discussed in more detail in the first discussion session. Solutions, and conceptual difficulties, can then be addressed in the second discussion session assigned to this problem set. Having someone in each group who is computationally skilled, could also be valuable for solving the problem sets.

Problem 1: Saturation Vapor Pressure

Recall the definition of the Gibbs potential

\[ G = H - TS \]

where \( H = U + pV \) denotes the enthalpy, and \( S \) the entropy. For systems at a fixed pressure and temperature it follows that

\[ 0 \leq -dG. \]

This leads to the minimum principle for the Gibbs potential, which states that the equilibrium state of a system in diathermal contact with a heat reservoir, and maintained at a fixed pressure, is the one that minimizes the Gibbs potential.

1. Prove that the minimum principle implies that for an equilibrium between two phases of a substance the specific potential of each phase has to be equal.

2. The minimum principle of the Gibbs potential can be used to derive the Clapeyron equation that describes the coexistence curves between two phases of matter. Use this principle to show that the Clapeyron equation for liquid water (subscript \( l \)) and water vapor (subscript \( v \)) takes the form:

\[ \frac{dp}{dT} = \frac{\ell}{T(v_v - v_l)} \]

where \( p \) is the saturation vapor pressure, and \( \ell \) is the specific enthalpy of vaporization\(^1\), defined such that \( \ell = h_v - h_l \).

3. The Clapeyron-Clausius approximation consists of neglecting \( v_l \) in comparison to \( v_v \) and replacing the latter with its expression for an ideal gas, i.e., \( v_v = R_v T/p \), so that

\[ \frac{d \ln p}{dT} = \frac{\ell}{R_v T^2}. \]

Compare how \( p \) changes with temperature by integrating the above equation (numerically) and comparing with more exact expressions such as that by Flatau, et al.\(^2\). Note that the Kirchhoff relation specifies that \( \ell(T) = \ell_0 + (c_{p,v} - c_{p,l})(T - T_0) \) where \( c_p \) denotes the isobaric specific heat. How does this improve the fit?

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\(^1\)This is often called the latent heat of vaporization and denoted by \( L \). I use the lower case following the convention for specific (or intensive) quantities (where \( T \) is an exception), and the more formal definition in terms of enthalpy.

4. Given that the saturation vapor pressure over ice and liquid should be equal at the triple point (for water this is at 273.16 K, or 0.1°C), from the above deduce which is larger as temperature decreases below the triple point.

**Problem 2: Lapse Rates**  The adiabatic lapse rate,

\[ \gamma \equiv -\frac{dT}{dz} \]

describes how temperature changes with height assuming that height and pressure changes are related hydrostatically. The adiabatic lapse rates for dry and saturated air are readily derived by recognizing that the moist static energy, \( \phi = c_p T + \ell v + gz \) is invariant for adiabatic transformations in which pressure changes hydrostatically\(^3\).

1. Calculate the dry lapse rate using the fact that \( d\phi = 0 \) for an adiabatic process.

2. Calculate the saturated lapse rate in a similar way. That is by noting that \( d\phi = 0 \) also for an adiabatic process allowing condensation. In this case \( d\phi = 0 \) although \( q_v \), the specific humidity of vapor changes with that of temperature, following its saturation value in accordance with the Clapeyron-Clausius relation. (Be careful to account for compositional effects on \( c_p \) and the dependence of \( \ell \) on \( T \).)

3. What is the dew-point lapse rate?

**Problem 3: Making Clouds**  The above principles can be used to calculate some useful information about the transformations one expects in association with vertical displacements of air parcels. For the following, first provide estimates for your answers assuming that the lapse-rates and dew-point lapse rates can be approximated by a mean value at some temperature appropriate to the problem, then repeat (if time allows) allowing for variations with temperature.

1. For an air-parcel at sea-level with a temperature of 300 K and a humidity of 80% calculate how high it must be lifted to reach saturation.

2. Assuming that condensation acts to keep the humidity at saturation, what would be the liquid water specific humidity for further adiabatic ascent of 2 km.

3. Repeat the above calculation for an air-parcel whose initial temperature is 10 K colder and whose initial relative humidity is only 60%.

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\(^3\)The moist static energy is often given the symbol \( h \) or \( h_m \), but I prefer \( \phi \) to avoid confusion with the enthalpy.