1 Theory

One way to measure the ability of the atmosphere to do work in accelerating parcels of buoyant air is to consider the vertical momentum equation in the form:

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\rho'}{\rho} g - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w.
\]  

(1)

In steady state, neglecting pressure, viscous and accelerations associated with the lateral \((x, y)\) advection of lower momentum air this equation reduces to the simple balance

\[
\frac{w}{\partial z} = -\frac{\rho'}{\rho} g.
\]  

(2)

Integrating yields an equation for the profile of the vertical component of the specific kinetic energy:

\[
\frac{w^2}{2} = \int_0^z w \frac{\partial w}{\partial z} dz = \int_0^z -\frac{\rho'}{\rho} g dz.
\]  

(3)

The rhs (right-hand-side) of (3) must have units of specific energy. It can be thought as an upper bound (because it neglects potentially dissipative forces such as viscosity, adverse pressure gradients and the entrainment of less vigorous air) on the energy a parcel achieves when displaced from rest to some height \(z\). As such it is often called the convective available potential energy or CAPE.

If we play with the rhs of (3) a little, by using the hydrostatic relation to change our integral to one in pressure we can write the CAPE as follows:

\[
\text{CAPE} = \int_0^z \frac{\rho'}{\rho} g dz = \int_{p_0}^{p_z} \frac{\rho'}{\rho} \frac{dp}{p} = \int_{p_0}^{p_z} \frac{\rho'}{\rho} R_d T_v \frac{dp}{p}
\]  

(4)

\[
= \int_{p_0}^{p_z} \frac{\rho'}{\rho} R_d T_v d\ln p
\]  

(5)

Where we have made use of the moist equation of state \(p = \rho R_d T_v\). From this same equation of state we note that as small perturbation in pressure are related to perturbations in virtual temperature and density:

\[p' = \rho' R_d T_v + \rho R_d T'_v\]  

(6)

dividing both sides by \(p\)

\[
\frac{p'}{p} = \frac{\rho'}{\rho} + \frac{T'_v}{T_v}
\]  

(7)

Again assuming that pressure perturbations are negligible this implies that density perturbations are compensated by perturbations in the virtual temperature such that

\[
\frac{\rho'}{\rho} = -\frac{T'_v}{T_v}
\]  

(8)

and hence that the CAPE is effectively given as:

\[
\text{CAPE} = R_d \int_{p_0}^{p_z} T'_v d\ln p
\]  

(9)
Where we have reversed the limits of integration to accommodate the sign. Note that this integral sums the difference between the virtual temperature of a parcel and that of the environment over $\ln p$. As such it is proportional (by approximately a factor of $R_d$) to the area between a saturated adiabat (which actually measures a saturated parcel’s temperature as it rises adiabatically) and the environmental temperature on a $\ln p$ Skew-$T$ diagram. Note that the graphical method of estimating the CAPE (from a thermodynamic diagram) neglects virtual effects, but often these are small.

CAPE measures the convective energy available to parcels. For dry air rising in an atmosphere whose temperature decreased at a rate more rapidly then they dry adiabatic lapse rate, CAPE would be measured by the area between the dry adiabat and the environmental lapse rate. Whether dry or saturated one can think of CAPE as measuring the work the atmosphere can do on parcels. The work is done because the parcels with an excess of $T_v$ relative to the environment necessarily have pressure forces at their base which are unbalanced, and hence experience vertical accelerations (the atmosphere squeezes it upwards). In this sense we can think of the atmosphere doing work on the parcel. In any real atmosphere the amount of energy available to convection is rarely realized because real parcels will be of some finite size and their accelerations will be balanced by pressure gradient forces (which arise from the need to move the air ahead of them out of the way) not to mention viscous forces and some dilution of the parcel (which tends to lower its $T_v$ perturbation) due to mixing with the environment as it accelerates upward. As such using CAPE to determine a velocity scale such that
\[
 w_{\text{max}} = 2\sqrt{\text{CAPE}} \tag{10}
\]
places an upper bound on the maximum vertical velocity a buoyant parcel.

## 2 Exercise

1. Using the attached two soundings, plot the moist static energy and the saturation moist static energy versus $\ln(p)$. Identify important regions of the soundings, i.e., pronounced inversions, hydrolapse (a sharp decrease in humidity), CAPE, CIN, moist layers, dry layers, etc. Compare and contrast the two soundings. How much CAPE is apparent in each sounding?

2. Take 5-10 soundings from around the equatorial belt and compare the average temperature at 500 hPa of these equatorial soundings versus 10 soundings taken near the mid-latitudes (45-50N). Apart from the soundings being much warmer along the equatorial belt do you notice any other difference based on the 500 hPa sounding itself.

3. Project: Compute the heat budget of the ITCZ regions using Reanalysis data.